

WITNESSING DP-RANK

ITAY KAPLAN AND PIERRE SIMON

ABSTRACT. We prove that in NTP_2 theories if p is a dependent type with $\text{dp-rank} > \kappa$, then this can be witnessed by indiscernible sequences of tuples satisfying p . If p has dp-rank infinity, then this can be witnessed by singletons (in any theory).

1. INTRODUCTION

In this note we answer a question of Alf Onshuus and Alexander Usvyatsov, whether dp-minimality can be witnessed by indiscernible sequences of singletons. We prove two general theorems regarding dp-rank .

Definition 1.1. Let $p(x)$ be a partial (consistent) type over a set A (x is a finite tuple, here and throughout the paper). We define the dp-rank of $p(x)$ (which is a cardinal or ∞) as follows:

- The dp-rank of $p(x)$ is always greater than or equal to 0. Let κ be a cardinal. We will say that $p(x)$ has $\text{dp-rank} \leq \kappa$ (which we write $\text{rk-dp}(p) \leq \kappa$) if given any realization a of p and any $1 + \kappa$ mutually indiscernible sequences over A , at least one of them is indiscernible over Aa .
- We say that p has $\text{dp-rank} \kappa$ over A (or $\text{rk-dp}(p) = \kappa$) if it has $\text{dp-rank} \leq \kappa$, but it is not the case that it has $\text{dp-rank} \leq \mu$ for any $\mu < \kappa$.
- We call p *dp-minimal* if it has $\text{dp-rank} 1$.
- We call p *dependent* if $\text{rk-dp}(p) < \infty$. This is equivalent to $\text{rk-dp}(p) \leq |T|^+$ (see Corollary 2.3).

Remark 1.2. It is easy to see that the set A does not matter, as long as p is defined over it. Indeed, for a set B over which p is defined, let us define for the sake of discussion $\text{rk-dp}(p, B)$ as the dp-rank of p over B similarly to the definition above but we add the requirement that the sequences are indiscernible over B . If $A \subseteq B$, and p is a type over A , then it is easy to see that $\text{rk-dp}(p, B) \leq \text{rk-dp}(p, A)$ while the other direction uses a standard application of Ramsey Theorem, so $\text{rk-dp}(p, B) = \text{rk-dp}(p, A)$.

Note also that if $q(x)$ extends $p(x)$ then $\text{rk-dp}(p(x)) \geq \text{rk-dp}(q(x))$, so:

Remark 1.3. Any extension of a dependent type is dependent.

Recall:

Definition 1.4. A (complete, first order) theory T is *dp-minimal* if the type $\{x = x\}$ is dp-minimal. The theory T is *dependent* if the type $\{x = x\}$ is dependent.

Dp-rank was originally defined in [OU11b] (the definition we give here is the same as in [KOU11]) and dp-minimality was first defined in [OU11a]. Both are simplifications of the various ranks appearing in [She]. It was shown in [Sim11] that the original definition of dp-minimality is equivalent to the definition given here.

Examples of dp-minimal theories include all o-minimal theories and C-minimal theories.

Note that the sequences that witness $\text{rk-dp}(p) > \kappa$ in Definition 1.1 can always be taken to be sequences of finite tuples, but can we bound the length?

Question. (*A. Onshuus, A. Usvyatsov*) Can we assume in the definition of dp-minimality that the indiscernible sequences are sequences of singletons?

We provide a positive answer in Corollary 1.7 below, but we need to add parameters to the base.

We prove the following two theorems:

Main Theorem A. *If p is a type over A which is independent (i.e. $\text{rk-dp}(p) = \infty$), then there is some $A' \supseteq A$ such that $|A' \setminus A|$ is finite, a realization $a \models p$ and A' -mutually indiscernible sequences of singletons $\langle I_i \mid i < |T|^+ + |A|^+ \rangle$ such that I_i is not indiscernible over $A'a$ for all i .*

From this we will deduce:

Corollary 1.5. *To check whether a theory is dependent it is enough to check that for every indiscernible sequence of singletons $\langle a_i \mid i < |T|^+ \rangle$ over some finite A , and for every singleton c , there is $\alpha < |T|^+$ such that $\langle a_i \mid i > \alpha \rangle$ is indiscernible over Ac .*

The second result is about dependent types, but to prove it we need to assume that the theory is NTP_2 .

Definition 1.6. A theory T is NTP_2 (does not have the tree property of the second kind) if there is no formula $\varphi(x, y)$ and array $\langle a_{i,j} \mid i, j < \omega \rangle$ such that for every $i < \omega$, $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is k -inconsistent (i.e. each subset of size k is inconsistent) and for every $\eta : \omega \rightarrow \omega$, the set $\{\varphi(x, a_{i,\eta(i)}) \mid i < \omega\}$ is consistent.

The class of NTP_2 theories contains both simple and dependent theories.

Main Theorem B. *Assume T is NTP_2 , and that p is a dependent type over A with $\text{rk-dp}(p) > \kappa$. Then there is some $A' \supseteq A$, some $a \models p$ and A' -mutually indiscernible sequences $\{I_i \mid i < 1 + \kappa\}$ such that each of them is not indiscernible over $A'a$ and all tuples in each I_i satisfy p .*

Now we can answer Question 1:

Corollary 1.7. *If T is not dp-minimal, then there is some finite set A , some singleton a and two A -mutually indiscernible sequences $\{I, J\}$ of singletons such that both I and J are not indiscernible over Aa .*

Proof. Right to left is obvious. For the other direction, if T is dependent then we may use Main Theorem B (since there are only two sequences, we may assume that A is finite). But if T is not dependent, then by Main Theorem A there exists such a , A and infinitely many such sequences. \square

The following question remains open:

Question. (*J. Ramakrishnan*) *Can we assume in the definition of dp-rank that the indiscernible sequences are sequences of singletons by adding parameters to the base?*

Our results show that this is indeed the case when the type is independent or the type of a singleton in an NTP_2 theory.

In Section 2 we prove Main Theorem A, and in Section 3 we prove Main Theorem B.

In Section 4 we provide an example that shows that the extra parameters are needed in Main Theorem B.

Throughout the paper, \mathfrak{C} will denote a monster model of the theory T (i.e. a very big saturated model).

2. ON DEPENDENT TYPES AND A PROOF OF MAIN THEOREM A

2.1. On dependent types. We start with the following easy observation, with a very straightforward proof.

Claim 2.1. Suppose $p(x)$ is a partial type over A . Then the following are equivalent:

- (1) There is $a \models p$ and A -mutually indiscernible sequences $\langle I_i \mid i < \omega \rangle$ such that the sequence $\langle I_i \mid i < \omega \rangle$ is indiscernible over Aa , and for each i , I_i is not indiscernible over Aa .
- (2) p is independent.
- (3) $\text{rk-dp}(p) > |T|^+ + |A|^+$.
- (4) There is an A -indiscernible sequence $\langle a_i \mid i < \omega \rangle$ such that $a_i \models p$, a formula $\varphi(x, y)$ and some c such that $\varphi(a_i, c)$ holds iff i is even.
- (5) There is an A -indiscernible sequence $\langle b_i \mid i < \omega \rangle$, a formula $\psi(y, x)$ and some $d \models p$ such that $\psi(b_i, d)$ holds iff i is even.
- (6) There is a set $\{a_i \mid i < \omega\}$ of realizations of p and a formula $\varphi(x, y)$ such that for every $s \subseteq \omega$, there is some c_s such that $\varphi(a_i, c_s)$ holds iff $i \in s$.
- (7) There is a set $\{b_i \mid i < \omega\}$ and a formula $\psi(y, x)$ such that for every $s \subseteq \omega$, there is some $d_s \models p$ such that $\psi(b_i, d_s)$ holds iff $i \in s$.

Proof. (1) implies (2) and (2) implies (3) are easy. Assume (3) and show (1). We can find $a \models p$ and A -mutually indiscernible sequences $\langle I_i \mid i < |T|^+ + |A|^+ \rangle$ such that for all i , I_i is not indiscernible over Aa . We may assume that the order type of these sequences is ω . The fact that I_i is not indiscernible over Aa is witnessed by some formula over A and increasing tuples from I_i , so we may assume that for infinitely many i , the formula is the same, and the position of these tuples does not depend on i (maybe changing a). Then, by Ramsey and compactness, we may assume that $\langle I_i \mid i < \omega \rangle$ is indiscernible over Aa .

(5) follows from (1): Denote $I_i = \langle a_{i,j} \mid j < \omega \rangle$. There is a formula $\psi(x, y)$ over A and an increasing tuple $k_0 < \dots < k_{n-1} < r_0 < \dots < r_{n-1}$ such that, letting $a_{i,\bar{k}} = (a_{i,k_0}, \dots, a_{i,k_{n-1}})$ (and similarly we define $a_{i,\bar{r}}$), $\psi(a_{i,\bar{k}}, a) \wedge \neg\psi(a_{i,\bar{r}}, a)$ holds for all $i < \omega$. The sequence $\langle b_i \mid i < \omega \rangle$ defined by $b_i = a_{i,\bar{k}}$ when i is even and $b_i = a_{i,\bar{r}}$ when i is odd satisfies (5). The fact that ψ is over A is no problem — we can add the parameters to b_i .

(2) follows from (5) is easy by compactness.

(6) is equivalent to (4) and (7) is equivalent to (5) by a standard application of Ramsey.

(6) follows from (5): By indiscernibility, we may extend $\langle b_i \mid i < \omega \rangle$ to $\langle b_r \mid r \in \mathcal{P}(\omega) \rangle$ (with some ordering), and so, for every subset $s \subseteq \mathcal{P}(\omega)$, there is some $d_s \models p$ such that $\psi(b_r, d_s)$ iff $r \in s$. For $i < \omega$, let $d_i = d_{\{r \subseteq \omega : i \in r\}}$. Then for each subset $r \subseteq \omega$, $\psi(b_r, d_i)$ iff $i \in r$. This gives us (6). The same exact argument gives that (7) follows from (4). \square

Proposition 2.2. *If p is a dependent type over A , then there is $B \subseteq A$ of size $|B| \leq |T|$ such that $p|_B$ is dependent.*

Proof. By Claim 2.1 (6), it cannot be that there exists a formula $\varphi(x, y)$ and a set $\{a_i \mid i < \omega\}$ or realizations of p such that for each $s \subseteq \omega$, there is some c_s such that $\varphi(a_i, c_s)$ holds iff $i \in s$. By compactness, there is no formula $\varphi(x, y)$ such that for all finite $B \subseteq A$ we can find such a set $\{a_i \mid i < \omega\}$ of realizations of $p|_B$ and such c_s for $s \subseteq \omega$. So for each formula $\varphi(x, y)$, there is some finite $B_\varphi \subseteq A$ such that there is no such set. Let $B = \bigcup_\varphi B_\varphi$. Then $p|_B$ is easily seen to be dependent. \square

Corollary 2.3. *The following are equivalent for a type $p(x)$ over A :*

- (1) $p(x)$ is independent.
- (2) $\text{rk-dp}(p) > |T|^+$.

Proof. If p is dependent, then there is some $B \subseteq A$ such that $p|_B$ is dependent and $|B| \leq |T|$. By Claim 2.1 (3), this means that $\text{rk-dp}(p|_B) \leq |T|^+$, so $\text{rk-dp}(p) \leq |T|^+$. \square

In this section we show that some useful properties that are true in dependent theories are actually true in the local context as well.

Fact 2.4. [KOU11, Theorem 4.11] *If p is a dependent type over A , and $a_i \models p$ for $i < n < \omega$, then $\text{tp}(a_0, \dots, a_{n-1}/A)$ is also dependent.*

Recall the notions of forking and dividing. All the definitions and properties we need can be found in [CK12].

Proposition 2.5. *If p is dependent type over a model M , and q is a global non-forking extension of p (i.e. an extension to \mathfrak{C}), then q is invariant over M .*

Proof. Suppose that $\varphi(x, c_0) \wedge \neg\varphi(x, c_1) \in q$ where $c_0 \equiv_M c_1$. Then using a standard technique, we can assume that c_0, c_1 start an indiscernible sequence $\langle c_0, c_1, \dots \rangle$ over M . The set $p(x) \cup \left\{ \varphi(x, c_i)^{(i \text{ is even})} \mid i < \omega \right\}$ is inconsistent by Claim 2.1. This means that for some formula $\psi(x) \in p$, $\{\psi(x) \wedge \varphi(x, c_{2i}) \wedge \neg\varphi(x, c_{2i+1}) \mid i < \omega\}$ is inconsistent, and so $\psi(x) \wedge \varphi(x, c_0) \wedge \neg\varphi(x, c_1)$ divides over M — contradiction. \square

Proposition 2.6. (*shrinking of indiscernibles*) *Suppose $p(x)$ is a dependent type over A and that B is a set of realizations of p .*

If $I = \langle a_i \mid i < |T|^+ + |B|^+ \rangle$ is an A -indiscernible sequence, then some end-segment is indiscernible over AB . Note that the size of A and the size of the tuple a_i do not matter.

Proof. We may assume that B is finite because $\text{tp}(B/A)$ is dependent iff $\text{tp}(B_0/A)$ is dependent for every finite $B_0 \subseteq B$. The type $\text{tp}(B/A)$ is dependent by Fact 2.4. The proof easily follows from Corollary 2.3. \square

2.2. Proof of Main Theorem A.

Definition 2.7. Let $n < \omega$. Let $p(x)$ be a type over A . we say that p is 1-independent over A if there is a realization $a \models p$ and A -mutually indiscernible sequences $\langle I_i \mid i < \omega \rangle$ of singletons such that the sequence $\langle I_i \mid i < \omega \rangle$ is indiscernible over Aa and for each $i < \omega$, I_i is not indiscernible over Aa .

We say that p is 1-dependent over A if it is not 1-independent over A . We say that p is 1-dependent if it is 1-dependent over any $A' \supseteq A$ such that $A' \setminus A$ is finite.

Observe that by Claim 2.1, if $p(x)$ is dependent then it is 1-dependent. Also, as in Remark 1.2, this definition does not depend on A .

Claim 2.8. If $p(x)$ is a type over A which is 1-dependent, then:

- For every $A' \supseteq A$ such that $A' \setminus A$ is finite, every A' -indiscernible sequence $\langle a_i \mid i < |T|^+ + |A|^+ \rangle$ of tuples satisfying p and singleton c , there is some $\alpha < |T|^+ + |A|^+$ such that the end-segment $\langle a_i \mid \alpha < i \rangle$ is indiscernible over $A'c$.

Proof. To simplify notation, assume $A = A' = \emptyset$. Towards contradiction we find a formula $\varphi(\bar{x}, y)$ and an indiscernible sequence $\langle \bar{a}_i \mid i < \omega \rangle$ such that \bar{a}_i is a tuple of length n of tuples satisfying p and $\varphi(\bar{a}_i, c)$ holds iff i is even. By the proof of Claim 2.1 (i.e. (5) implies (4), with $p = \text{tp}(c)$), there is an indiscernible sequence $\langle c_{\bar{i}} \mid \bar{i} \in \omega^{n+1} \rangle$ (ordered lexicographically) of singletons such that $\varphi(\bar{a}_0, c_{\bar{i}})$ holds iff the last number in \bar{i} is even. We may also assume (by Ramsey) that the sequence $\langle \bar{c}_{\bar{i}} \mid \bar{i} \in \omega^n \rangle$ is indiscernible over \bar{a}_0 , where $\bar{c}_{\bar{i}} = \langle c_{\bar{i} \smallfrown j} \mid j < \omega \rangle$.

Suppose $\bar{a}_0 = (a_{0,0}, \dots, a_{0,n-1})$ where $a_{0,i} \models p$. Since p is 1-dependent over \emptyset , there is some $i_0 < \omega$ such that $\langle c_{i_0 \smallfrown \bar{i}} \mid \bar{i} \in \omega^n \rangle$ is indiscernible over $a_{0,0}$. By assumption, p is 1-dependent over $a_{0,0}$. Inductively, we can find $i_1, \dots, i_{n-1} < \omega$ such that $\bar{c}_{(i_0, \dots, i_{n-1})}$ is indiscernible over \bar{a}_0 — contradiction. \square

The following theorem is a reformulation of Main Theorem A:

Theorem 2.9. *If $p(x)$ is a type over A which satisfies the conclusion of Claim 2.8, then it is dependent.*

Proof. Again, assume $A = \emptyset$. Suppose p is a counterexample. By Claim 2.1, there is an indiscernible sequence $\langle a_i \mid i < |T|^+ \rangle$ such that $a_i \models p$, a formula $\varphi(x, y)$ and some tuple $c = (c_0, \dots, c_{n-1})$ such that $\varphi(a_i, c)$ holds iff i is even. By assumption, there is some end-segment which is indiscernible over c_0 . Applying the conclusion of Claim 2.8 again with $A' = \{c_0\}$, we get an end-segment which is indiscernible over $c_0 c_1$. Continuing like this, we get an end-segment which is indiscernible over c — contradiction. \square

Since dependent implies 1-dependent, we get:

Corollary 2.10. *The type $p(x)$ is 1-dependent iff it is dependent iff it satisfies the conclusion of Claim 2.8.*

Corollary 1.5 follows:

Corollary 2.11. *A theory T is dependent iff for every indiscernible sequence of singletons $\langle a_i \mid i < |T|^+ \rangle$ over some finite A , and for every singleton c , there is $\alpha < |T|^+$ such that $\langle a_i \mid \alpha < i \rangle$ is indiscernible over Ac .*

Proof. Apply Corollary 2.10 with $p(x) = \{x = x\}$. \square

3. PROOF OF MAIN THEOREM B

3.1. Preliminaries on NTP_2 theories. From here to the end of the section, we assume that the theory is NTP_2 .

In the study of forking in NTP_2 theories, it is sometime useful to consider independence relations. For instance, we denote $a \perp_B^f C$ for $\text{tp}(a/BC)$ does not fork over B . Similarly, $a \perp_B^i C$

means that there is a global extension (i.e. an extension to \mathfrak{C}) of $\text{tp}(a/BC)$ which is Lascar invariant over B , meaning that if d and c have the same Lascar strong type over B then either both $\varphi(x, c)$ and $\varphi(x, d)$ are in this extension or neither of them is. We do not really need Lascar strong type for this section, because we only work over models. Over a model, Lascar invariance is the same as invariance.

In the proof we shall only use the following facts about NTP_2 theories. These were proved in [CK12].

Definition 3.1. (strict independence) We say that $\text{tp}(a/Bb)$ is strictly invariant over B (denoted by $a \downarrow_B^{\text{ist}} b$) if there is a global extension p , which is Lascar invariant over B (so $a \downarrow_B^i b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \downarrow_B^f c$.

Fact 3.2. In NTP_2 theories

- (1) *Forking equals dividing over models.*
- (2) *“Kim’s Lemma”: If $\varphi(x, a)$ divides over A , and $\langle b_i \mid i < \omega \rangle$ is a sequence satisfying $b_i \equiv_A a$ and $b_i \downarrow_A^{\text{ist}} b_{<i}$, then $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent. In particular, if $\langle b_i \mid i < \omega \rangle$ is an indiscernible sequence then it witnesses dividing of $\varphi(x, a)$.*

Recall:

Definition 3.3. Suppose p is a global type which is invariant over a set A .

- (1) We say that a sequence $\langle a_i \mid i < \alpha \rangle$ is a Morley sequence of a type p over $B \supseteq A$ if $a_0 \models p|_B$ and for all $i < \alpha$, $a_i \models p|_{Ba_{<i}}$. This is an indiscernible sequence over B .
- (2) We let the type $p^{(\alpha)}$ be the union of $\text{tp}(\langle a_i \mid i < \alpha \rangle / B)$ running over all $B \supseteq A$. This is again an A -invariant type.
- (3) If q is also an A -invariant global type, we define $p \otimes q$ as the union of $\text{tp}(a, b/B)$ running over all $B \supseteq A$ where $a \models p|_B$ and $b \models q|_{Ba}$. This is also an A -invariant global type.
- (4) Similarly, given a sequence $\langle p_i \mid i < \alpha \rangle$ of A -invariant global types, we define $\bigotimes_{i < \alpha} p_i$ as the union of $\text{tp}(\langle a_i \mid i < \alpha \rangle / B)$ running over all $B \supseteq A$, where $a_i \models p_i|_{Ba_{<i}}$. Again, this is an A -invariant global type.

In the definition above, all types may have infinitely many variables.

Remark 3.4. If $\{J_0, \dots, J_k\}$ is a set of mutually indiscernible sequences over $C \supseteq A$, and $\langle a_i \mid i < \alpha \rangle$ is a Morley sequence of a global A -invariant type over $\{J_0, \dots, J_k\} \cup C$ then $\{J_0, \dots, J_k, \langle a_i \mid i < \alpha \rangle\}$ is mutually indiscernible over C .

We also need to recall the notions of heir and coheir:

Definition 3.5. A global type $p(x)$ is called a coheir over a set A , if it is finitely satisfiable in A . Note that in this case, it is invariant over A , and $p^{(\alpha)}$ is also a coheir over A .

It is called an heir over A if for every formula over A , $\varphi(x, b) \in p$, there exists some $a' \in A$ such that $\varphi(x, a') \in p$.

Claim 3.6. If p is an A -invariant global type and $p^{(\omega)}$ is both an heir and a coheir over A , then any Morley sequence of p over A , $\langle a_i \mid i < \omega \rangle$ satisfies $a_{\geq i} \downarrow_A^{\text{ist}} a_{< i}$ for any $i < \omega$.

Proof. The type $p^{(\omega)}$ is a global A -invariant (so also A -Lascar invariant) type that extends $\text{tp}(a_{\geq i}/Aa_{< i})$, and if $c \models p^{(\omega)}|AC$ then $\text{tp}(C/AC)$ is finitely satisfiable over A (since $p^{(\omega)}$ is an heir over A), and it follows that $C \downarrow_A^f c$. \square

Claim 3.7. Given any A -invariant global type $p(x)$, we can find a model $M \supseteq A$ such that p is an heir over M .

Proof. Construct inductively a sequence of models M_i for $i < \omega$. Let M_0 be any model containing A . Let $M_{i+1} \supseteq M_i$ be such that for every formula $\varphi(x, y)$ over M_i , if $\varphi(x, a) \in p$ then there is some such a in M_{i+1} . Finally, let $M = \bigcup_{i < \omega} M_i$. \square

Lemma 3.8. *Let M be a model. Suppose that p is an M -invariant global type such that $p^{(\omega)}$ is an heir-coheir over M . Suppose I is an endless Morley sequence of p over M . If I is indiscernible over Ma then $\text{tp}(a/MI)$ does not fork over M .*

Proof. By Fact 3.2, it is enough to see that the type does not divide over M . Suppose $\varphi(x, b_0) \in \text{tp}(a/MI)$ divides over M , where φ is over M and $b_0 \subseteq I$. For $i \geq 1$ choose tuples $b_i \subseteq I$ of the same length as b_0 that appear after b_0 in increasing order. By Claim 3.6, $b_i \downarrow_M^{\text{ist}} b_{< i}$ so by “Kim’s lemma” (Fact 3.2), it must witness dividing. But this is a contradiction to the fact that I is indiscernible over Ma . \square

3.2. Proof of the main theorem. The following is the key definition in the proof.

Definition 3.9. Let

- (1) p be a global A -invariant type such that $p|_A$ is dependent (we call such types A -invariant and A -dependent),
- (2) B some set containing A ,
- (3) $\varphi(x, y)$ a formula over A ,
- (4) and a a tuple of length $\text{lg}(y)$.

Then we define $\text{alt}(\varphi, B, a, p)$ to be the maximal number n such that there is a realization $\langle a_i \mid i < n \rangle \models p^{(n)}|_B$, such that $\varphi(a_i, a)$ alternates for $i < n$, i.e. such that $\varphi(a_i, a) \Leftrightarrow \neg \varphi(a_{i+1}, a)$ for $i < n - 1$.

Note that $\text{alt}(\varphi, B, a, p)$ exists by Claim 2.1 (4). Observe that $\text{alt}(\varphi, B, a, p) \geq \text{alt}(\varphi, B', a, p)$ when $B' \supseteq B \supseteq A$, but not necessarily the other way. Sometimes there is equality:

Lemma 3.10. *Suppose p is a global A -invariant and A -dependent type, a some tuple and I an indiscernible sequence over Aa .*

Then: for every infinite subset $I' \subseteq I$ and for any formula $\varphi(x, y)$ over A , $\text{alt}(\varphi, IA, a, p) = \text{alt}(\varphi, I'A, a, p)$.

Proof. Obviously, $\text{alt}(\varphi, IA, a, p) \leq \text{alt}(\varphi, I'A, a, p)$.

Conversely, suppose we have some n such that $\bar{a} = \langle a_i \mid i < n \rangle \models p^{(n)}|_{I'A}$ alternates as in the definition. Let $\bar{x} = (x_0, \dots, x_{n-1})$. We want to show that the type

$$p^{(n)}(\bar{x})|_{IA} \cup \left\{ \varphi(x_i, a)^{(\text{if } \varphi(a_i, a))} \mid i < n \right\}$$

is consistent.

Take any finite subset and write it as $\psi(\bar{x}, b, c) \wedge \xi(\bar{x}, a)$ where $b \subseteq I$, $c \subseteq A$. As I' is infinite, and as I is indiscernible over Aa we can find $b' \in I'$ such that $b' \equiv_{Aa} b$ so $\mathfrak{C} \models \exists \bar{x} \psi(\bar{x}, b', c) \wedge \xi(\bar{x}, a)$ iff $\mathfrak{C} \models \exists \bar{x} \psi(\bar{x}, b, c) \wedge \xi(\bar{x}, a)$. Now, $\psi(\bar{x}, b, c) \in p^{(n)}$, and $p^{(n)}$ is A invariant, hence $\psi(\bar{x}, b', c) \in p^{(n)}$, and since \bar{a} satisfies $\psi(\bar{x}, b', c) \wedge \xi(\bar{x}, a)$, we are done. \square

We will deduce Main Theorem B from the following theorem:

Theorem 3.11. *Suppose $p(x)$ is a dependent type over C with $\text{rk-dp}(p) > \kappa$. Assume this is witnessed by $c \models p$ and $\{I_i \mid i < 1 + \kappa\}$ where I_i has order type ω for $i < 1 + \kappa$.*

Then there are

- $C' \supseteq C$ with $|C' \setminus C|$ finite, $c' \models p$ and J_0

such that

- $\{J_0\} \cup \{I_i \mid i < 1 + \kappa\}$ are mutually indiscernible over C' ; $c' \equiv_{C \cup \{I_i \mid 0 < i < 1 + \kappa\}} c$; J_0 is not indiscernible over $C'c'$ and
- all the tuples in J_0 satisfy p .

Proof. Denote $I_i = \langle f_{i,j} \mid j < \omega \rangle$. By compactness, we can find $f_{i,j}$ for $j \in \mathbb{Z}$ and $i < 1 + \kappa$ such that, letting $I'_i = \langle f_{i,j} \mid j \in \mathbb{Z}, j < 0 \rangle$, $\{I'_i \frown I_i \mid i < 1 + \kappa\}$ are mutually indiscernible over C . Let U be a non-principal ultrafilter on ω . For $i < 1 + \kappa$, let p_i be global coheir over I'_i defined by:

- for a formula $\psi(z, y)$ and a tuple $a \in \mathfrak{C}$, $\psi(z, a) \in p_i$ iff $\{n < \omega \mid \models \psi(f_{i,-n}, a)\} \in U$.

So each p_i is invariant over $\bigcup_{i < 1 + \kappa} I'_i$ and we can consider the type $\left(\bigotimes_{0 < i < 1 + \kappa} p_i^{(\omega)} \right)^{(\omega)}$, and find a model $M \supseteq C \cup \bigcup_{i < 1 + \kappa} (I'_i \frown I_i)$ such that this type is an heir over M (using Claim 3.7).

Let $\langle K_i \mid i < 1 + \kappa \rangle \models \bigotimes_{i < 1 + \kappa} p_i^{(\omega)}|_M$, then:

- each K_i is a Morley sequence of p_i over M ,
- since $\{I'_i \frown I_i \mid i < 1 + \kappa\}$ are mutually indiscernible over C , and p_i are finitely satisfiable in I'_i , $\langle K_i \mid i < 1 + \kappa \rangle \equiv_C \langle I_i \mid i < 1 + \kappa \rangle$ (this follows from the fact that the order type of I'_i is $\omega^* - \omega$ in reverse), and

- by 3.4, $\{K_i \mid i < 1 + \kappa\}$ are mutually indiscernible over M .

By the second bullet, there is an automorphism of \mathfrak{C} that fixes C (but may move M and p_i) and maps $\langle K_i \mid i < 1 + \kappa \rangle$ to $\langle I_i \mid i < 1 + \kappa \rangle$. By applying it we may assume that $\langle K_i \mid i < 1 + \kappa \rangle = \langle I_i \mid i < 1 + \kappa \rangle$.

Let $\mu = |T|^+$. Let $J = \langle d_i \mid i < \mu \rangle$ be a Morley sequence of $\left(\bigotimes_{0 < i < 1 + \kappa} p_i^{(\omega)}\right)$ over MI_0 so that d_0 is the infinite tuple $\langle I_i \mid 0 < i < 1 + \kappa \rangle$. Note that I_0 is still indiscernible over JM , J is indiscernible over I_0M and $\{I_i \mid i < 1 + \kappa\}$ are mutually indiscernible over $M \cup \{d_i \mid 0 < i < \mu\}$ (by Remark 3.4).

Now, I_0 is not indiscernible over Cc . So there are increasing tuples a_0 and a_1 from I_0 of the same length and a formula $\varphi(x, y)$ over C such that $\neg\varphi(c, a_0) \wedge \varphi(c, a_1)$ holds. By indiscernibility, there is an automorphism σ of \mathfrak{C} that fixes JM and takes a_0 to a_1 . Let $c_0 = c$ and $c_1 = \sigma(c_0)$. Then $\varphi(c_0, a_1) \wedge \neg\varphi(c_1, a_1)$ holds.

By Proposition 2.6, for some $\alpha < \mu$, the sequence $J' = \langle d_i \mid \alpha < i < \mu \rangle$ is indiscernible over $M_{c_0c_1}$. By Lemma 3.8, $c_0c_1 \downarrow_M^f J'$.

Let $r(x)$ be a global non-forking (over M) extension of $\text{tp}(c_0/MJ')$ ($= \text{tp}(c_1/MJ')$). Since $\text{tp}(c_0/C)$ is dependent, $r(x)$ is M -dependent and M -invariant (by 2.5). Let $n = \text{alt}(\varphi, MJ, a_1, r)$, and let $\langle e_i \mid i < \omega \rangle$ be a Morley sequence of r over MJ that witnesses this, i.e. such that $\varphi(e_i, a_1)$ alternates n times. Let $\langle e_i \mid \omega \leq i < \omega + \omega \rangle$ be a Morley sequence of r over $MJ_{c_0c_1e_{<\omega}}$.

So:

- $I'_0 = \langle e_i \mid i < \omega + \omega \rangle$ is an MJ -indiscernible sequence, and moreover $\{I_i \mid 0 < i < 1 + \kappa\} \cup \{I'_0\}$ is a set of MJ' -mutually indiscernible sequences.

Now, both $\langle c_0, e_\omega, e_{\omega+1}, \dots \rangle$, $\langle c_1, e_\omega, e_{\omega+1}, \dots \rangle$ are Morley sequences of r over MJ' . But if in addition $\langle c_0, e_0, e_1, \dots \rangle$ and $\langle c_1, e_0, e_1, \dots \rangle$ are also Morley sequences of r over MJ' , then since one of c_0, c_1 , adds an alternation of the truth value of $\varphi(x, a_1)$, this is a contradiction to the choice of e_i and to Lemma 3.10 (which we can use because J is indiscernible over a_1 , and J' is infinite). Let $c' \in \{c_0, c_1\}$ such that $\langle c', e_\omega, e_{\omega+1}, \dots \rangle$ is not a Morley sequence of r over MJ' . Note that $c' \equiv_{MJ} c$ and that MJ contains $C \cup \{I_i \mid 0 < i < 1 + \kappa\}$.

So, $\langle c', e_0, \dots \rangle \not\equiv_{MJ'} \langle c', e_\omega, \dots \rangle$ and hence the sequence I'_0 is not indiscernible over $c'MJ'$. Let J_0 be some infinite subset of I'_0 of order type ω that witnesses this, and let $C' \supseteq C$ be such that $|C' \setminus C|$ is finite, and $C' \subseteq MJ'$ so that J_0 is not indiscernible over $C'c'$.

It is now easy to check that all conditions are satisfied. \square

Now let us conclude:

Proof. (of Main Theorem B) Suppose p is a dependent type over A with $\text{rk-dp}(p) > \kappa$. Consider the family \mathcal{F} of triples (s, c, J, A') such that

- $c \models p$, $s \subseteq 1 + \kappa$, $J = \langle I_i \mid i < 1 + \kappa \rangle$; $A \subseteq A'$; J is a sequence of A' -mutually indiscernible sequences such that for each $i < 1 + \kappa$, I_i is not indiscernible over $A'c$; all tuples in I_i for $i \in s$ realize p .

By assumption, \mathcal{F} is not empty. Define the following order relation on these triples:

- $(s, c, J, A') \leq (s', c', J', A'')$ iff $(s \subseteq s', A' \subseteq A'', \text{ if } i \in s \cup ((1 + \kappa') \setminus s') \text{ then } I_i = I'_i \text{ and } c' \equiv_{A' \cup \{I_i \mid i \in s \cup ((1 + \kappa') \setminus s')\}} c)$.

It is easy to see that by compactness \mathcal{F} satisfies the conditions of Zorn's Lemma, so it has a maximal member (s_0, c_0, J_0, A'_0) . By Theorem 3.11, $s_0 = 1 + \kappa$ and we are done. \square

4. AN EXAMPLE

Here we describe an example that shows why we need to add parameters in the proof of Main Theorem B.

Let $L = \{P, Q, <, \wedge, f\}$ where P, Q are unary predicates, $<$ is a binary predicate and \wedge, f are binary function symbols. Let T^\forall be the following universal L -theory:

- $(P, <, \wedge)$ is a tree where \wedge is the meet-function (a tree is a partial order such that for every element x , the set $\{a \mid a < x\}$ is linearly ordered, and $x \wedge y = \max\{a \mid a < x \ \& \ a < y\}$; its existence is part of the theory).
- $P \cap Q = \emptyset$.
- $\forall x (P(x) \vee Q(x))$.
- f is a binary function from $\{x, y \in P \mid x < y\}$ to Q such that $x < y < z \Rightarrow f(x, y) = f(x, z)$.

Note that T^\forall has the amalgamation property and the joint embedding property (and the structures generated by n elements have at most n^2 elements). So, by Fraïssé's theorem it has a model completion T . So T eliminates quantifiers (see [Hod93, Theorem 7.4.1]) and is ω -categorical.

Claim 4.1. The theory T is dependent. Moreover, $\text{rk-dp}(\{x = x\}) = 2$.

Proof. First we show $\text{rk-dp}(\{x = x\}) \leq 2$.

By Main theorems A and B, it is enough to check that if a is an singleton and I, J, K are mutually indiscernible sequences of singletons over some set A , then one of them is indiscernible over Aa .

Assume $a \in Q$. Note that if I is in Q , then for I to be not indiscernible over Aa , a needs to be in I . In that case, since J is not indiscernible over Aa , it is not indiscernible over AI — contradiction. This means that none of I, J, K is in Q . Since I is not indiscernible over Aa , there is a term $t(x)$ over A and increasing tuples a_0, a_1 of the same length such that $t(a_0) = a$ and $t(a_1) \neq a$. There is also such a term $s(y)$ and tuples b_0, b_1 for J , so $t(a_0) = s(b_0)$ and so

$t(a_1) = s(b_0)$ — contradiction. Note that this shows that in this case there cannot even be two such sequences.

Assume $a \in P$. Note that if one of I, J, K , say K , is an A -indiscernible sequence in Q , then there is a term $t(x)$ over A such that $t(a) \in K$ (K is not constant). So at most one of the sequences are in Q — at least two sequences are in P . Without loss these are I, J . The theory of trees is dp-minimal (see [Sim11], or observe that our tree $(P, <, \wedge)$ has elimination of quantifiers which makes it very easy to prove), so we may assume I is indiscernible in the tree language over Aa . It is not hard to see that by the choice of f and quantifier elimination, I is still indiscernible over Aa even in the extended language: there are only five possible types of non-constant indiscernible sequences in P , so by dividing into cases the result follows.

To show equality, we find two mutually indiscernible sequences I, J over a singleton c and some a such that both I and J are not indiscernible over ac . Let $c \in P$, and let $I = \{a_i \mid i < \omega\} \subseteq P$ be an increasing sequence (i.e. $a_i < a_j$ whenever $i < j$) below c . Let J be some non-constant sequence in Q . By Ramsey, we choose them so that I, J are mutually indiscernible over c . Let $a \in P$ be some singleton such that $a_0 < a < a_1$, and $f(a, c) \in J$. \square

Claim 4.2. We cannot find two \emptyset -mutually indiscernible sequences of singletons I, J and a singleton a such that both I and J are not indiscernible over a .

Proof. If $a \in Q$, then by the proof of Claim 4.1 above there cannot be two such sequences. If $a \in P$, it cannot be that both sequences are in P (because then since the tree $(P, < \wedge)$ is dp-minimal, one of them is indiscernible over a in the tree language and also in the extended language by the choice of f). Assume $J \subseteq Q$. By quantifier elimination J must be indiscernible over a — contradiction. \square

REFERENCES

- [CK12] Artem Chernikov and Itay Kaplan. Forking and dividing in NTP_2 theories. *Journal of Symbolic Logic*, 77(1):1–20, 2012.
- [Hod93] Wilfrid Hodges. *Model Theory*, volume 42 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, Great Britain, 1993.
- [KOU11] Itay Kaplan, Alf Onshuus, and Alexander Usvyatsov. Additivity of the dp-rank. *Accepted for publication in the Transactions of the American Mathematical Society*, 2011. [arXiv:1109.1601](#).
- [OU11a] Alf Onshuus and Alexander Usvyatsov. On dp-minimality, strong dependence and weight. *Journal of symbolic logic*, 76(3):737–758, 2011.
- [OU11b] Alf Onshuus and Alexander Usvyatsov. Thorn orthogonality and domination in unstable theories. *Fundamenta Mathematicae*, 214(3):241–268, 2011.
- [She] Saharon Shelah. Strongly dependent theories. *submitted*.
- [Sim11] Pierre Simon. On dp-minimal ordered structures. *Journal of Symbolic Logic*, 76(2):448–460, 2011.